ε-factors for $GL(2)$ ~ Didier

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Question: Existence of L-functions for $GL(2)$?, Is it possible to calculate ε-factors for these L-functions?

§I, L-functions (Godement - Jacquet):

Notations: Let F = local + non-arch, with p = uniformizer.

$$G = GL_2(F)$$
$$A = M_2(F) \supseteq M = M_2(\mathcal{O}_F)$$
(π, V) smooth irrep.

Harmonic Analyses:

$$C_c^\infty(A) = \left\{ \langle \mathbb{1}_{a+p^j M} \rangle : a \in A, j \in \mathbb{Z} \right\}$$

Take:

$$\psi \in \hat{F} \setminus \{1\}, \quad \psi_A = \psi \circ \mathrm{tr} \quad \text{(additive)}.$$

Fourier:
$$\hat{\phi}(x) = \int_A \phi(y)\, \psi_A(xy)\, dy$$

$$\hat{\hat{\phi}}(x) = \phi(-x) \quad \text{(for a good choice of measure)}.$$

$q = \# k_F$.
$$\mu_\psi(M) = q^{2\ell(\psi)}, \quad \ell(\psi) \text{ is the level of } \psi.$$

$$\mu_{\psi_a} = \|a\|_F^2 \mu_\psi.$$

Zeta integrals:

Recall
$$\xi^*(s) = \mathcal{M}\left(\mathbb{1}_{\mathcal{O}_F} \times e^{-x^2/2}, s\right).$$
$$L(s,f) = \mathcal{M}(f|_{i\mathbb{R}}, s).$$

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Def^n: $\zeta(\phi, f, s) = \int_G \phi(x) f(x) \|\det x\|^s d^\times x$, $\phi \in C_c^\infty(A)$

$f \in \mathcal{E}(\pi)$

$\mathcal{E}(\pi) = \mathrm{Vect}(g \mapsto \langle v, \pi(g) v' \rangle$ where $v, v' \in V_\pi)$.

Thm: (Godement - Jacquet) (π, V) irrep smooth

1) $\zeta(\phi, f, s)$ converges for $\mathrm{Re}(s) \gg 1$, uniformly in ϕ, f.

2) $\zeta(\phi, f, s) \in \mathbb{C}(q^{-s})$.

3) $Z(\pi) := \{ \zeta(\phi, f, s + \frac{1}{2}) : \phi, f \}$ ideal of $\mathbb{C}[q^s, q^{-s}]$, and $\exists P_\pi \in \mathbb{C}[X]$, $P_\pi(0) = 1$, such that $Z(\pi) = P_\pi(q^{-s})^{-1} \mathbb{C}[q^{\pm s}]$

we write $L(s, \pi) = P_\pi(q^{-s})^{-1}$.

4) There exist $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$ s.t.

$$\zeta(\hat{\phi}, \check{f}, \tfrac{3}{2} - s) = \gamma(\pi, s, \psi)\, \zeta(\phi, f, \tfrac{1}{2} + s), \ \forall \phi, f$$

$f(g^{-1})$.

Corollary: Let $\varepsilon(\pi, s, \psi) := \gamma(\pi, s, \psi) \dfrac{L(\pi, s)}{L(\check{\pi}, 1-s)}$

$$= \frac{\zeta(\hat{\phi}, \check{f}, \frac{3}{2} - s) / L(\check{\pi}, 1-s)}{\zeta(\phi, f, s + \frac{1}{2}) / L(\pi, s)}$$

1) Then $\varepsilon(\pi, s, \psi)\, \varepsilon(\check{\pi}, 1-s, \psi) = \omega_\pi(-1)$

2) Et $\varepsilon(\pi, s, \psi) = a q^{bs}$ for certain a, b.

Proof: 1) $\xi(\hat{\hat{\Phi}}, f, \tfrac{1}{2}+s) \underset{\substack{E.F.\\(\hat\Phi, \check f, 1-s)}}{=} \gamma(\check\pi, 1-s, \psi)\, \xi(\hat\Phi, \check f, \tfrac{3}{2}-s)$.

Fourier $\|$

$$\underline{\underline{\omega_\pi(-1)}}\, \xi(\Phi, f, \tfrac{1}{2}+s) \underset{\substack{E.F.\\(\Phi, f, s)}}{=} \underline{\underline{\gamma(\check\pi, 1-s, \psi)\gamma(\pi, s, \psi)}}\, \xi(\Phi, f, \tfrac{1}{2}+s)$$

2) $L(s,\pi) = \sum \xi(\Phi_i, f_i, s)$

$$\varepsilon(s,\pi,\psi) = L(\check\pi, 1-s)^{-1} \gamma(s,\pi,\gamma) L(s,\pi)$$

$$= L(\check\pi, 1-s)^{-1} \sum \underbrace{\xi(\Phi_i, f_i, s)\gamma(s,\pi,\gamma)}_{\substack{\| \, E.F. \\ \xi(\Phi_i, \check f_i, \frac{3}{2}-s)}}$$

$$\in \mathbb{C}[q^s, q^{-s}]$$

$\varepsilon(1-s, \check\pi, \psi)$ also (similar)

So $\varepsilon(s,\pi,\psi) \in \mathbb{C}[q^s, q^{-s}]^*$ ~~Hen~~. Hence,

$\varepsilon(s,\pi,\psi) = a q^{bs}$, for some a, b.

$\varepsilon(\pi, s, \psi) = \varepsilon(\pi, \tfrac{1}{2}, \psi) q^{n(\pi,\psi)(\frac{1}{2}-s)}$.

Formal series: (indexed by the valuation).

$$G_m = \{x \in G : v_F(\det x) = m\}$$

$$Z_m(\Phi, f) := \int_{G_m} \Phi(x) f(x)\, d^*x = \int_G \Phi_m(f)$$

$$Z(\Phi, f, X) := \sum_{m\in\mathbb{Z}} z_m(\Phi, f) X^m.$$

<u>Proposition</u>: $Z(\pi)$ is a $\mathbb{C}[X,X^{-1}]$-module containing $\mathbb{C}[X,X^{-1}]$

. If $f_0 \in C(\pi)\setminus\{0\}$ lib $Z(\Phi, f_0, q^{-1/2}X)$ suffisant.

<u>Preuve</u>: $(g,h)\cdot\Phi(x) := \Phi(g^{-1}xh)$

$$\widehat{(g,h)\Phi} = \|\det(gh^{-1})\|^2\,(h,g)\cdot\hat{\Phi}$$

$$Z((g,h)\Phi,(g,h)f,X) = X^{v_F(\det gh^{-1})}\,Z(\Phi,f,X)$$

hence module.

Choose f, $\Phi = \mathbb{1}_K$ where f is $K\times K$-inv

$\longrightarrow Z(\Phi,f,X) =$ constant $\neq 0$.

§II. ε-factors for cuspidal-π:

(Gauss sums).

<u>Tools</u>: (Representations of $GL_2(F)$)

- (π,V) : $\mathrm{ind}_{G_E}^{G_F}\tau \longleftrightarrow$ cuspidal $(V_N=0)$ $\quad$ τ : $(G_E\to\mathbb{C}^\times)$

$(p\neq 2)$

$$\chi\left(\mathbb{1}_2\times\begin{pmatrix}0&1\\0&0\end{pmatrix}\right) \longleftrightarrow \chi\times St_G.$$

$$\chi\left(\mathbb{1}_2\times\begin{pmatrix}0&0\\0&0\end{pmatrix}\right) \longleftrightarrow \chi\circ\det$$

$$\chi_1\oplus\chi_2 \longleftrightarrow \mathrm{ind}_B^G(\chi_1\oplus\chi_2),\ \chi\in\hat{T}\ ($$

- $\pi \longleftrightarrow (A, \Xi)$ "Strata" s.t. $\pi = c\text{-}\mathrm{ind}_{K_A}^{G}(\Xi)$

$\mathcal{A}$ = order

Ξ rep of $K_{\mathcal{A}} := \{g \in G : g^{-1}\mathcal{A} g = \mathcal{A}\}$

$U_A^n = 1 + \mathcal{B}^n$, $\mathcal{B} = \mathrm{rad}(\mathcal{A})$

Def^n: $T(A, \Xi, \psi) := \sum_{x \in U_A / U_A^{n+1}} \check{\Xi}(cx)\, \psi_A(cx)$ where $c\mathcal{A} = \mathcal{B}^{-n}$

"Gauss sums"

$$= \underbrace{\tau(\Xi, \psi)}_{\in \mathbb{C}^{\times}} 1_{\check{W}} .$$

Thm: $(\pi, V) \longrightarrow (A, \Xi)$, level $\psi = 1$,

$$m = \ell_A(\Xi) = \min(n : U_A^{n+1} \subset \ker \Xi)$$

$$= e_A\, \ell(\pi)$$

Then $\varepsilon(\pi, s, \psi) = q^{2\ell(\pi)(\frac{1}{2}-s)}\, \tau(\Xi, \psi)\, (\mathcal{A} : \mathcal{B}^{n+1})^{-1/2}$

Proof: $K = \mathcal{U}_A^{n+1}$, $\Phi = \mathbb{1}_K / \mu^*(K)$

$$\zeta(\Phi, \pi, s) := \int_G \Phi(x)\, \pi(x)\, \|\det\|^s\, d^*x$$

Recall (spectral projections)

- If $e_K = \frac{1}{\mu^*(K)} \mathbb{1}_K$, $\pi(e_K) : V \longrightarrow V^K$ projection.
- If $e_\xi(x) = \frac{1}{\mu^*(K)} \dim(\xi)\, \mathrm{tr}(\xi(x^{-1}))$

$\pi(e_\xi) : V \longrightarrow V^{\xi}$.

$$\xi := \Xi|_{U_A}, \quad V^{\xi} = V^{\Xi}.$$

$$\pi(e_\xi)\,\pi(g)\,\pi(e_\xi)\,v = \begin{cases} \Xi(g)\,v & \text{if } v \in K_A \\ 0 & \text{if not} \end{cases}.$$

$$\pi(e_\xi)\,\xi(\phi, f, s)\,\pi(e_\xi) \underset{\text{formal}}{\overset{①}{=}} \xi(\underbrace{e_\xi * \phi * e_\xi}_{\psi}, \pi, s)$$

② ‖

$$\pi(e_\xi) \quad \underline{\quad CCL \quad} \quad \xi(e_\xi * \phi * e_\xi, \pi, s) = \pi(e_\xi).$$

$$\xi(\psi, \pi, s) \; \varepsilon(\psi, \pi, s) \underset{CCL}{=} \pi(e_\xi)\,\varepsilon(\psi, \pi, s)$$

$$\underset{EF.}{=} \xi(\hat{\psi}, \check{\pi}, 2-s)$$

$$= \int_K \underbrace{\hat{\psi}(x)}\,\check{\pi}(x)\,\|x\|^{2-s}\,d^*x$$

$$= \text{calc } \check{e}_\xi \circledast \hat{\phi} * \check{e}_\xi$$

$$= \int_K \hat{\phi}(x)\,\underbrace{\check{\pi}(\check{e}_\xi)\,\check{\pi}(x)\,\check{\pi}(\check{e}_\xi)}_{\check{\Xi}(x)\,\pi(\check{e}_\xi) \text{ by } (*)}\,\|x\|^{2-s}\,d^*x$$

<u>Calcul of $\hat{\phi}$</u>: $\quad K = U_A^{n+1} = 1 + \mathcal{B}^{n+1}$

$$\hat{\phi}(x) = \int_G \phi(y)\,\psi_A(xy)\,dy. \quad \operatorname{supp} \phi = \mathcal{B}^{-n}$$

$$\underset{\operatorname{supp}\phi = K}{=} \frac{1}{\mu^n(K)} \int_K \psi_A(xy)\,dy \qquad x \in \mathcal{B}^{-n}$$

$\overline{\overline{\varepsilon(\psi)}} = 1 \quad \frac{1}{\mu^*(K)} \int_{\mathcal{B}^{n+1}} \psi_A(x(1+y))\,dy$

$\downarrow$

$y \in 1+\mathcal{B}^{n+1}$

$$= \frac{\psi_a(x)}{\mu^*(K)} \int_{\mathcal{B}^{n+1}} \underbrace{\psi_A(xy)}_{\equiv 1}\,dy = \boxed{\psi_a(x)\,\frac{\mu(\mathcal{B}^{n+1})}{\mu^*(K)}\,\hat{\xi}(x)}$$

$$\varepsilon = \frac{\mu(\mathcal{B}^{n+1})}{\mu^*(K)} \int_{\substack{K = 1+\mathcal{B}^{n+1} \\ = CA}} \psi_A(x)\,\check{\Xi}(x)\,\|x\|^{2-s}\,dx^*$$

$$\underset{cv}{=} \frac{\cdots}{\cdots} \int \psi_A(cx)\,\check{\Xi}(cx)\,\|cx\|^{2-s}\,d^*x$$

$$\underset{\substack{\text{inv par} \\ U_A^{n+1}}}{=} \frac{1}{\cdots} \underbrace{\sum_{x \in U_A/U_A^{n+1}} \psi_A(cx)\,\check{\Xi}(cx)}_{\tau(\Xi,\psi)\,\pi(\rho_\xi)} \underbrace{\int_{y \in U_A^{n+1}} \psi_A(cy)\,\check{\Xi}(cy)\,\|cy\|^{2-s}\,dx}_{\text{only depend on } A,\ \mathcal{B}.\ \square}$$

where $\chi(1+x) = \mu(x)$, $n(\chi) > 2\ell(\pi)$

Formulas:

Rep.	$\varepsilon(s,\pi,\psi)$
$(*,\Xi)$ unif	$q^{2\ell(\pi)(\frac{1}{2}-s)}\,(A,\mathcal{B}^{n+1})^{-1/2}\,\tau((\Xi),\psi)$
$\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$	$\varepsilon(\chi_1,s,\psi)\,\varepsilon(\chi_2,s,\psi)$
$\chi \circ \det_G$	$\varepsilon(\chi,s-\frac{1}{2},\psi)\,\varepsilon(\chi,s+\frac{1}{2},\psi) = \Phi(\phi)^{-2} q^{2s-1}$
$\chi \times St_G$	$\varepsilon(\chi,s,\psi)$