Representations of the Weil group 2

§ Abstract Machinery : Let

G = profinite group

K_0G = Grothendieck group of G (i.e. the free abelian group of symbols $[\rho]$, where $\rho \in \mathrm{irrep}_{sm}(G)$). a.k.a group of virtual representations.

Remark : we view the set of iso. classes of finite dimensional smooth representations of G in K_0G as following : if ρ is such a rep.

$$\rho = \underbrace{\rho_1 \oplus \cdots \oplus \rho_r}_{\text{irr.}} \,, \text{ to obtain an element}$$

$$[\rho] = \sum_{i=1}^{r} [\rho_i] \in K_0G$$

Abuse of notation : drop the brackets & write $\rho = [\rho]$.

$\exists$ dim map $K_0G \longrightarrow \mathbb{Z}$ defined in the natural way.

Define :
$$\tilde{K}_0G := \bigcup_{\substack{H \leq G \\ \text{open}}} K_0H$$

and denote its elements as pairs (H, ρ), where $\underset{\text{open}}{H \leq G}$ & $\rho \in K_0H$.

$$\Gamma(G) := \mathrm{Hom}(G, \mathbb{C}^\times) \subseteq K_0G$$

so
$$\tilde{\Gamma}(G) := \bigcup_{\substack{H \leq G \\ \text{open}}} \Gamma(H) \subseteq \tilde{K}_0G.$$

Remark: 1) If G' is another profinite group s.t. $G' \to G$,

then $\tilde{K}_0G \subseteq \tilde{K}_0G'$

2) $G = \varprojlim_i G_i$, then $\tilde{K}_0G = \bigcup_i \tilde{K}_0G_i$

Enables us to reduce to finite case.

Def^n (Induction constants): Let A be an abelian group, & G = profinite group.

a) An induction constant on G (with values in A) is a function

$$\mathcal{F} : \tilde{K}_0 G \to A$$

such that

i) for each open $H \leq G$, the map $\mathcal{F}|_{K_0 H}$ is a group homo.

ii) if $H \subset J$ are open subgroups of G & $(H, \rho) \in K_0 H$ has dimension 0, then

$$\mathcal{F}(J, \mathrm{Ind}_H^J \rho) = \mathcal{F}(H, \rho)$$

b) A division on G (with values in A) is a function

$$D : \tilde{\Gamma}(G) \to A$$

Remarks: An induction constant $\mathcal{F}$ give rise to a division $\partial\mathcal{F}$ via restriction and $\partial\mathcal{F}$ is called the boundary of $\mathcal{F}$

Def[n]: A division D on G is called pre-inductive on G if

$$D = \partial\mathcal{F}$$

for some induction constant $\mathcal{F}$ on G.

Lemma 1: An induction constant is completely determined by its boundary $\partial\mathcal{F}$. $\mathrm{Im}(\mathcal{F})$ is contained in the abelian group generated by the values of $\partial\mathcal{F}$.

Proof. Exercise using Brauer induction theorem.

- We can reduce to G finite
- Let ρ be an irreducible rep of $H \subset G$ of dim m
- $$[\rho] - m[1_H] = \sum_{1 \leq i \leq n} \mathrm{Ind}_{H_i}^H ([\chi_i] - [1_{H_i}])$$

for various $(H_i, \chi_i) \in \tilde{\Gamma}(H)$.

$$\mathcal{F}(H, \rho) = \partial\mathcal{F}(H, 1_H)^m \prod_{1 \leq i \leq n} \partial\mathcal{F}(H_i, \chi_i)\partial\mathcal{F}(H_i, 1_{H_i})^{-1}$$

Remark. $(H,\rho) \in \tilde{K}_0 G$. Let 1_H = trivial character on H.

Put $R_{G/H} = \mathrm{Ind}_H^G 1_H$.

Then $\mathcal{F}([\rho] - m[1_H]) = \mathcal{F}([\mathrm{Ind}_H^G \rho] - m[R_{G/H}])$,

where $m = \dim \rho$.

Lemma 2. G = pro finite group and let D be a division on G. Suppose there is a family $\mathcal{H}$ of open normal subgroups H of G such that

a) the canonical map $G \to \varprojlim_{H \in \mathcal{H}} G/H$ is an isomorphism, and

b) the restriction $D_{G/H}$ of D to $\tilde{\Gamma}(G/H)$ is pre-inductive on G/H $\forall\, H \in \mathcal{H}$.

The division D is pre-inductive on G. If $\mathcal{F}$ is an induction constant on G with boundary D, then $D_{G/H}$ is the boundary of $\mathcal{F}|_{\tilde{K}_0(G/H)}$.

§ Main statement : Existence of the local constants

Let E/F be a finite separable extension.

For $\varphi \in \hat{F}$, we set $\varphi_E = \varphi \circ \mathrm{Tr}_{E/F} \in \hat{E}$.

Recall from Ilyana's talk $R_n^{ss}(F)$ = isomorphism class of semi-simple smooth representations of W_F of dimension n.

$R_n^{o}(F)$ = ——————— irreducible

Write $R^{ss}(F) = \bigcup_{n \geq 1} R_n^{ss}(F)$ & $R^{o}(F) = \bigcup_{n \geq 1} R_n^{o}(F)$.

Theorem A: Let $\varphi \in \hat{F}$, $\varphi \neq 1$ & $\bar{F} \supset E \supset F$, E/F is finite.

There is a unique family of functions:

$$R^{ss}(E) \longrightarrow \mathbb{C}[q^s, q^{-s}]^\times$$
$$\rho \longmapsto \varepsilon(\rho, s, \varphi_E).$$

with the following properties:

1) If χ is a character of $E^\times$, then:

$$\varepsilon(\chi \circ \alpha_E, s, \psi_E) = \varepsilon(\chi, s, \psi_E)$$

where $\alpha_E : W_E \to E^\times$ is the Artin reciprocity map.

2) If $\rho_1, \rho_2 \in R^{ss}(E)$, then

$$\varepsilon(\rho_1 \oplus \rho_2, s, \psi_E) = \varepsilon(\rho_1, s, \psi_E) \cdot \varepsilon(\rho_2, s, \psi_E).$$

3) If $\rho \in R_n^{ss}(E)$ and $E \supset K \supset F$, then

$\mathrm{Ind}_{E|K} = \mathrm{Ind}_{W_E}^{W_K}$
$R_{E|K} = \mathrm{Ind}_{W_E}^{W_K} 1_E$

$$\frac{\varepsilon(\mathrm{Ind}_{E|K}\rho, s, \psi_K)}{\varepsilon(\rho, s, \psi_E)} = \frac{\varepsilon(R_{E|K}, s, \psi_K)^n}{\varepsilon(1_E, s, \psi_E)^n}$$

The quantity $\varepsilon(\rho, s, \psi)$, $\rho \in R^{ss}(F)$ is called the Langlands–Deligne local constant of ρ, relative to the character $\psi \in \hat{F}$ & complex variable $s \in \mathbb{C}$.

We enumerate some of its interesting properties:

Proposition A: Let $\psi \in \hat{F}$, $\psi \neq 1$ & $\rho \in R^{ss}(F)$. Then:

a) $\exists\ n(\rho, \psi) \in \mathbb{Z}$ s.t.

$$\varepsilon(\rho, s, \psi) = q^{n(\rho,\psi)(\frac{1}{2} - s)}\, \varepsilon(\rho, \tfrac{1}{2}, \psi)$$

b) Let $a \in F^\times$. Then:

$$\varepsilon(\rho, s, a\psi) = \det \rho(a)\ \|a\|^{\dim(\rho)\,(s - \frac{1}{2})}\, \varepsilon(\rho, s, \psi)$$

$$n(\rho, a\psi) = n(\rho, \psi) + v_F(a) \dim(\rho).$$

c) We have, moreover, a functional equation:

$$\varepsilon(\rho, s, \psi)\, \varepsilon(\check{\rho}, 1-s, \psi) = \det \rho(-1)$$

d) There is an integer n_ρ such that if χ is a character of $F^\times$ of level $k \geq n_\rho$, then

$$\varepsilon(\chi \otimes \rho, s, \psi) = \det \rho(c(\chi))^{-1} \varepsilon(\chi, s, \psi)^{\dim \rho},$$

for any $c(\chi) \in F^\times$ such that $\chi(1+x) = \psi(c(\chi)x)$, $x \in \mathfrak{p}^{[k/2]+1}$.

We will use the abstract machinery in the following context:

Let L/F be finite and Galois, put $G = Gal(L/F)$.

Now $\tilde{\Gamma}(G) = \bigcup_{H \leq G} \Gamma(H)$, but $H = Gal(L/E)$, for $E = L^H$

$$\Gamma(H) = Hom(H, \mathbb{C}^\times) = Hom(H^{ab}, \mathbb{C}^\times)$$

but $H^{ab} \simeq E^* / N_{L/E}(L^*)$ (by local class field theory).

So we think of $\tilde{\Gamma}(G)$ as the set of pairs (E, χ) where E ranges over fields between L & F, χ over characters of E^* which are null on $N_{L/E}(L^*)$.

Deligne's Global input

We assume the next result to prove theorem A & Prop. A.

Theorem B: Let L/F be a finite Galois extension with Galois group G. Then $\exists\, \psi \in \hat{F}$, $\psi \neq 1$ such that the following division on G

$$D_\psi^{L/F} : \tilde{\Gamma}(G) \longrightarrow \mathbb{C}[q^s, q^{-s}]^\times$$
$$(E, \chi) \longmapsto \varepsilon(\chi, s, \psi_E) \longrightarrow \text{Tate's local constant.}$$

is pre-inductive on $G = Gal(L/F)$.

§ Step 1 : removing restriction on φ.

Recall : if $\varphi : F \to \mathbb{C}^\times$ is a non-trivial character then all characters of F are of the form $\varphi_a(x) = \varphi(ax)$ for unique $a \in F^\times$.

By lemma 2 & theorem B, the division :

$$D_\varphi^{L/F} : (E, \chi) \longmapsto \varepsilon(\chi, s, \varphi_E)$$

is pre-inductive on

$$\Omega_F = \mathrm{Gal}(\bar{F}/F) = \varprojlim_{\substack{L/F \text{ finite} \\ \text{Galois}}} \mathrm{Gal}(L/F) = \varprojlim_{\substack{L/F \text{ finite} \\ \text{Galois}}} \Omega_F/\Omega_L$$

By lemma 1 we see that $D_\varphi^{L/F}$ is the boundary of the induction constant defined by

$$(\Omega_E, \rho) \longmapsto \varepsilon(\rho, s, \varphi_E).$$

Claim 1 : For $a \in F^\times$ the following function

$$\tilde{K}_0 \Omega_F \longrightarrow \mathbb{C}^\times$$

$$(E, \rho) \longmapsto \det \rho(a) \, \|a\|_E^{(s-\frac{1}{2})\dim \rho}$$

is an induction constant on Ω_F.

- ☆ The first property is clear to verify
- ☆ The second property follows from the "transfer theorem"

K/E is finite Galois

$$\begin{tikzcd} W_K^{ab} \arrow[r, "\simeq"] & K^\times \\ W_E^{ab} \arrow[u, "ver_{K/E}"] \arrow[r, "\simeq"] \arrow[ur, phantom, "\circlearrowright"] & E^\times \arrow[u, hook] \supset F^\times \end{tikzcd}$$

and the following : $\det \mathrm{Ind}_{K/E}\rho = \det\rho \circ ver_{K/E}$.

So we conclude that

$$(E,\rho) \longmapsto \det\rho(a)\, \|a\|_E^{(s-\frac{1}{2})\dim\rho}\, \varepsilon(\rho, s, \varphi_E)$$

is also an induction constant on Ω_F.

The boundary of this induction constant is :

$$(E,\chi) \mapsto \chi(a)\, \|a\|_E^{(s-\frac{1}{2})}\, \varepsilon(\chi, s, \varphi_E) = \varepsilon(\chi, s, a\varphi_E)$$

This division is pre-inductive , and the boundary of the induction constant $(E,\rho) \mapsto \varepsilon(\rho, s, a\varphi_E)$ (by definition).

Theorem B holds for all $\varphi \in \hat{F}$, $\varphi \neq 1$. This along with Lemma 2 finishes the proof of Theorem A for representations of Galois groups.

Now we prove Proposition A (1), (2) for representations of Galois groups. Recall that ε-factors of characters satisfy the following relation :

$$\boxed{\varepsilon(\chi, s, \varphi) = q^{(\frac{1}{2}-s)\, n(\chi,\varphi)}\, \varepsilon(\chi, \tfrac{1}{2}, \varphi)}$$

for some $n(\chi,\varphi) \in \mathbb{Z}$. Thus by Lemma 1 we conclude (1) .

To prove (2) we note the following :

$$\varepsilon(\rho, s, a\varphi) = q^{n(\rho, a\varphi)(\frac{1}{2}-s)}\, \varepsilon(\rho, \tfrac{1}{2}, a\varphi) \qquad (\text{by } (1))$$

$$\& \quad \varepsilon(\rho, s, a\varphi) = \det(\rho(a))\, \|a\|^{\dim(\rho)(s-\frac{1}{2})}\, q^{n(\rho,\varphi)(\frac{1}{2}-s)}\, \varepsilon(\rho, \tfrac{1}{2}, \varphi).$$

which implies (2) by comparison

§ Step 2 : Extend these results to reps of Weil gps.

Fix ϖ a uniformizer of F. Let $\phi \in \widehat{F}^\times$ be unramified. Write $\phi(\varpi) = q^{-s(\phi)}$, for some $s(\phi) \in \mathbb{C}$.

For a E/F finite, $\varpi_E \in E$ uniformizer, we also have

$$\phi_E(\varpi_E) = q_E^{-s(\phi)} = q^{-f_{E/F}\, s(\phi)}.$$

Thus if $\chi \in \widehat{E}^\times$, we have

$$\varepsilon(\chi\phi_E, s, \psi_E) = \varepsilon(\chi, s+s(\phi), \psi_E).$$ (Xavier's talk).

We want to extend this identity to representations:

Claim 2. Let $(\Omega_E, \rho) \in \widetilde{K}_0 \Omega_F$, let $\phi \in \widehat{F}^\times$ be unramified + of finite order. Then $\varepsilon(\phi_E \otimes \rho, s, \psi_E) = \varepsilon(\rho, s+s(\phi), \psi_E)$.

Proof: They are both induction constants with the same boundary, hence by lemma 1 they are equal.

Now let $\mathbb{1}_E$ be the trivial character of the Weil group $\mathcal{W}_E$ and define

$$\boxed{\lambda_{E/F}(s,\psi) = \frac{\varepsilon\left(\mathrm{Ind}_{W_E}^{W_F} \mathbb{1}_E, s, \psi\right)}{\varepsilon\left(\mathbb{1}_E, s, \psi_E\right)}}$$

Corollary : $\lambda_{E/F}(s,\psi)$ is constant in s.

Proof: Let $\phi \in \widehat{F}^\times$ unramified of finite order. We have

$$\phi \otimes \mathrm{Ind}_{W_E}^{W_F} \mathbb{1}_E \cong \mathrm{Ind}_{W_E}^{W_F} \phi_E$$

so $$\lambda_{E/F}(s,\varphi) = \frac{\varepsilon(\mathrm{Ind}_{W_E}^{W_F}\phi_E, s, \varphi)}{\varepsilon(\phi_E, s, \varphi_E)}$$

$$= \frac{\varepsilon(\mathrm{Ind}_{W_E}^{W_F}\mathbb{1}_E, s+s(\phi), \varphi)}{\varepsilon(\mathbb{1}_E, s+s(\phi), \varphi_E)}$$

$$= \lambda_{E/F}(s+s(\phi), \varphi_E)$$

Thus, $\lambda_{E/F}(s+s(\phi), \varphi) = \lambda_{E/F}(s,\varphi)$ for all unramified characters ϕ of finite order. That is $\lambda_{E/F}(s+\zeta, \varphi) = \lambda_{E/F}(s,\varphi)$ for all roots of unity $\zeta \in \mathbb{C}$.

From representations of Weil groups to reps. of Galois group.

Because of additivity its enough to consider irrep smooth of W_E. Let $\rho \in \mathrm{irrep}_{sm}(W_E)$.

Claim: there is an unramified character ϕ of W_F such that $\phi_E \otimes \rho$ factors through a representation ρ_0 of Ω_E.

Step 1: $\rho(I_E)$ is a finite subgroup of $GL(V)$.

Step 2: Conjugation action of Frobenius on the finite image $\rho(I_E)$ has finite order:

$\mathrm{Frob}_E \in W_E$ be a Frobenius, conjugation by Frob_E induces an isomorphism of I_E. Induces $\rho(\mathrm{Frob}_E)$ automorphism of $\rho(I_E)$, thus it is finite order:

$\rho(x) \longmapsto \rho(\mathrm{Frob}_E)\, \rho(x)\, \rho(\mathrm{Frob}_E)^{-1}$ has finite order.

So $\exists\, k \geq 1$ s.t. $\forall\, x \in I_E$ we have

$$\rho(\mathrm{Frob}_E)^k \rho(x) \rho(\mathrm{Frob}_E)^k = \rho(x).$$

So $\rho(\mathrm{Frob}_E)^k$ commutes with all $\rho(I_E)$.

<u>Step 3</u>: $\rho(\mathrm{Frob}_E)^k$ commutes with $\rho(W_E)$.

$$\rho(\mathrm{Frob}_E)^k \in \mathrm{Cent}(\rho(W_E)).$$

<u>Step 4</u>: Schur's lemma : $\rho(\mathrm{Frob}_E)^k$ is scalar.

Since ρ is irreducible, Schur's lemma says that any linear endomorphism of V that commutes with the whole image $\rho(W_E)$ is a scalar multiple of identity. Therefore $\exists\, c \in \mathbb{C}^\times$ with

$$\rho(\mathrm{Frob}_E)^k = c \cdot \mathrm{Id}_V.$$

<u>Step 5</u>: Choose an unramified character to kill the scalar c.

$$\chi : W_E \to \mathbb{C}^\times \quad \text{trivial on } I_E.$$

so $\chi(\mathrm{Frob}_E)$ determines χ.

we want

$$\chi(\mathrm{Frob}_E)^k = c^{-1}$$

so we take $\chi(\mathrm{Frob}_E) = k^{\text{th}}$ root of c^{-1}.

Step 6: $\chi \otimes \rho$ has finite image.

consider the twist $(\chi \otimes \rho)(g) = \chi(g)\rho(g)$

$$\chi(\mathrm{Frob}_E)^k \rho(\mathrm{Frob}_E)^k = c^{-1} \cdot c \cdot \mathrm{Id}_V = \mathrm{Id}_V.$$

Hence $(\chi \otimes \rho)(\mathrm{Frob}_E)$ has finite order dividing k.

$\Rightarrow (\chi \otimes \rho)(\mathcal{W}_E)$ is finite.

Step 7: Factorization through a finite quotient.

If the image of a continuous homo. $\mathcal{W}_E \to GL(V)$ is finite, then its kernel is open normal subgroup of $\mathcal{W}_E$ of finite index.

So $\exists$ K/E finite Galois s.t.

$$\mathcal{W}_E / \ker(\chi \otimes \varphi) \simeq \Omega_E / \Omega_K \simeq \mathrm{Gal}(K/E).$$

Thus we reduced it to the Galois case.

The character $\chi : \mathcal{W}_E \to \mathbb{C}^\times$ can be thought of as a character of $\chi : E^\times \longrightarrow \mathbb{C}^\times$ so χ is of the form

$\chi = \phi \circ N_{E/F}$. where ϕ is a character of $F^\times$.

Define:

to remove dependence on ϕ.

$$\boxed{\varepsilon(\rho, s, \psi_E) = \varepsilon(\rho_0, s - s(\phi), \psi_E)}$$

The first two properties of theorem A are easy to check. We verify:

3) If $\rho \in R_n^{ss}(E)$ and $E \supset K \supset F$, then

$$\frac{\varepsilon(\mathrm{Ind}_{E/K}\rho, s, \psi_K)}{\varepsilon(\rho, s, \psi_E)} = \frac{\varepsilon(R_{E/K}, s, \psi_K)^n}{\varepsilon(1_E, s, \psi_E)^n}$$

we know for representations of Galois group that this identity is true.

So if $\phi_E \otimes \rho$ factorises through a rep of Galois group then the same holds for $\phi_K \otimes \mathrm{Ind}_{E/K}\rho = \mathrm{Ind}_{E/K}\phi_E \otimes \rho$

so
$$\frac{\varepsilon(\mathrm{Ind}_{E/K}\rho, s, \psi_K)}{\varepsilon(\rho, s, \psi_E)} = \frac{\varepsilon(\mathrm{Ind}_{E/K}\rho_0, s - s(\phi), \psi_K)}{\varepsilon(\rho_0, s - s(\phi), \psi_E)}$$

$$= \frac{\varepsilon(R_{E/K}, s - s(\phi), \psi_K)^n}{\varepsilon(1_E, s - s(\phi), \psi_E)^n}$$

$$\underset{\text{constant in } s}{=} \frac{\varepsilon(R_{E/K}, s, \psi_K)^n}{\varepsilon(1_E, s, \psi_E)^n}$$

This finishes the proof theorem A.

Let us now prove proposition A for representations of Weil group.

3) Functional equation: for Galois case its clear because:

$$(\Omega_E, \tau) \mapsto \begin{cases} \varepsilon(\tau, s, \psi_E)\,\varepsilon(\check{\tau}, 1-s, \psi_E) \\ \det \tau(-1) \end{cases}$$

are both induction constants on $\tilde{K}_0 \Omega_F$ with boundary

$$(E, \phi) \mapsto \varepsilon(\phi, s, \psi_E)\, \varepsilon(\check{\phi}, 1-s, \psi_E) = \phi(-1)$$

Thus these two induction constants are the same.

It is enough to treat Galois case because of the definition of ε-factor.

Theorem B L/F finite Galois of non. arch. local fields. $G = \mathrm{Gal}(L/F)$.

$\exists$ $\mathbb{L}/\mathbb{F}$ of global fields and a non-archimedean place v_0 of $\mathbb{F}$ s.t.

1) $\exists!$ place u_0 of $\mathbb{L}$ over v_0 s.t.

$$\underset{\mathrm{tpo}}{\mathbb{L}_{u_0} \cong L} \quad \text{which induces} \quad \mathbb{F}_{v_0} \cong F.$$

2) $G \cong \mathrm{Gal}(\mathbb{L}/\mathbb{F})$ canonically.

so for any intermediate field E, $F \subset E \subset L$

there is a unique field $\mathbb{E}$, $\mathbb{F} \subseteq \mathbb{E} \subseteq \mathbb{L}$ with

closure E in $L = \mathbb{L}_{v_0}$.

Fix a non-trivial character Ψ of $\mathbb{A}_{\mathbb{F}}/\mathbb{F}$ and for $\mathbb{E}$ set $\Psi_{\mathbb{E}} = \Psi \circ \mathrm{Tr}_{\mathbb{E}/\mathbb{F}}$.

Put $\psi = \Psi_{v_0}$.

$$G = \mathrm{Gal}(L/F) \simeq \mathrm{Gal}(\mathbb{L}/\mathbb{F}) = \mathbb{G}$$

we have a bijection $\tilde{\Gamma}(G) \cong \tilde{\Gamma}(\mathbb{G})$

$$(E, \chi) \mapsto (\mathbb{E}, \chi_0)$$

χ_0 is a character of $\mathbb{A}_{\mathbb{E}}^*/\mathbb{E}^*$ trivial on norms $N_{\mathbb{L}/\mathbb{E}}(\mathbb{A}_{\mathbb{L}}^*/\mathbb{L}^*)$.

<u>Claim:</u> there exist a character α of $\mathbb{A}_{\mathbb{F}}^*/\mathbb{F}^*$ such that $\alpha_{\mathbb{E}} = \alpha \circ N_{\mathbb{E}/\mathbb{F}}$

$$\varepsilon(\chi_\omega \alpha_{\mathbb{E},\omega}, s\, \Psi_{\mathbb{E},\omega}) = \begin{cases} \chi_\omega(c_v) \\ \end{cases}$$